

Robust Hadamard gate for optical and ion trap holonomic quantum computers

V.I. Kuvshinov¹, A.V. Kuzmin²

Joint Institute for Power and Nuclear Research "Sosny"

220109 Krasina str., 99, Minsk, Belarus

1 - e-mail: v.kuvshinov@sosny.bas-net.by

2 - e-mails: avkuzmin@sosny.bas-net.by; avkuzmin@dragon.bas-net.by.

Abstract

We consider one possible implementation of Hadamard gate for optical and ion trap holonomic quantum computers. The expression for its fidelity determining the gate stability with respect to the errors in the single-mode squeezing parameter control is analytically derived. We demonstrate by means of this expression the cancellation of the squeezing control errors up to the fourth order on their magnitude.

Holonomic quantum computation exploiting non-abelian geometrical phases (holonomies) was primarily proposed in Ref. [1] and was further worked out in Ref. [2]. Various implementations of holonomic quantum computer (HQC) have been proposed recently. Namely, it was suggested to realize the HQC within quantum optics (optical HQC) [3]. Laser beams in a non-linear Kerr medium were used for this purpose. Two different sets of control devices could be used in this case. The first one consists of one and two mode displacing and squeezing devices. The second one includes SU(2) interferometers. Squeezing and displacement of the vibrational modes of the trapped ions were suggested to use for realization of HQC in Ref. [4]. This implementation of HQC is mathematically similar to the first embodiment of the optical HQC offered in Ref. [3]. Particularly, expressions for the adiabatic connection and holonomies are the same. Another proposed implementation of HQC was HQC with neutral atoms in cavity QED [5]. The coding space was spanned by the dark states of the atom trapped in a cavity. Dynamics of the system was governed by the generalized Λ -system Hamiltonian. Mathematically similar semiconductor-based implementation of HQC was proposed in Ref. [6]. One-qubit gates were realized in the framework of the same generalized Λ -system as in Ref. [5]. However its physical implementation exploits semiconductor excitons driven by sequences of laser pulses [6]. For the two-qubit gate the bi-excitonic shift was used. The generalized Λ -system with different Rabi frequencies parametrization was exploited recently for HQC implemented by Rf-SQUIDs coupled through a microwave cavity [7]. One more solid state implementation of HQC based on Stark effect was proposed in Ref. [8].

Quantum computers including HQC are analog-type devices. Thus unavoidable errors in the assignment of the classical control parameters lead to an errors in quantum gates and in the case when the tolerance of quantum computation is not large enough the computation fails. This obstacle (inaccuracy) is also related to the decoherence problem [9]. The effect of the

errors originated from the imperfect control of classical parameters was studied for \mathbf{CP}^n model of HQC in Ref. [10] where the control-not and Hadamard gates were particularly considered. Approach based on the non-abelian Stokes theorem [11] was proposed in our previous Letter [12]. Namely, the general expression for fidelity valid for arbitrary implementation of HQC in the case of the single control error having the arbitrary size was derived. Simple approximate formulae was found in the small error limit. Adiabatic dynamics of a quantum system coupled to a noisy classical control field was studied in Ref. [13]. It was demonstrated that stochastic phase shift arising in the off-diagonal elements of the system's density matrix can cause decoherence. The investigation of the robustness of non-abelian holonomic quantum gates with respect to parametric noise due to stochastic fluctuations of the control parameters was presented in Ref. [14]. In this work three stability regimes were discriminated for HQC model with logical qubits given by polarized excitonic states controlled by laser pulses. Noise cancellation effect for simple quantum systems was considered in Ref. [15]. The decoherence of HQC was discussed, for instance, in Refs. [16, 17]. Berry phase in classical fluctuating magnetic field was considered in Ref. [18]. Fidelity decay rates for HQC interacting with the stochastic environment were obtained recently in Ref. [19].

In this Letter we consider one-qubit gates for optical HQC and HQC on trapped ions. The mathematical model based on squeezing and displacing transformations of the qubit state is the same for both these implementations (compare [3] and [4]). We consider one possible implementation of Hadamard gate and analytically derive the expression for its fidelity determining the gate stability with respect to the errors in the single-mode squeezing parameter control. We demonstrate the cancellation of the control errors up to the fourth order on their magnitude.

Let us briefly remind some results concerning HQC. In holonomic quantum computer non-abelian geometric phases (holonomies) are used for implementation of unitary transformations (quantum gates) in the subspace C^N spanned on eigenvectors corresponding the degenerate eigenvalue of parametric isospectral family of Hamiltonians $F = \{H(\lambda) = U(\lambda)H_0U(\lambda)^+\}_{\lambda \in M}$, where $U(\lambda)$ is unitary [1]. The λ 's are the control parameters and M represents the space of the control parameters. The subspace C^N is called quantum code (N is the dimension of the degenerate computational subspace). Quantum gates are realized when the control parameters are adiabatically driven along the loops in the control manifold M . The unitary operator mapping the initial state vector into the final one has the form $\bigoplus_{l=1}^R e^{i\phi_l} \Gamma_\gamma(A_\mu^l)$, where l enumerates the energy levels of the system, ϕ_l is the dynamical phase, R is the number of different energy levels of the system under consideration and the holonomy associated with the loop $\gamma \in M$ is given by:

$$\Gamma_\gamma(A_\mu) = \hat{P} \exp \int_\gamma A_\mu d\lambda_\mu. \quad (1)$$

Here \hat{P} denotes the path ordering operator, A_μ is the matrix valued adiabatic connection given by the expression [20]:

$$(A_\mu)_{mn} = \langle \varphi_m | U^+ \frac{\partial}{\partial \lambda_\mu} U | \varphi_n \rangle, \quad (2)$$

where $|\varphi_k\rangle$, $k = \overline{1, N}$ are the constant basis vectors of the corresponding eigenspace C^N , index μ enumerates the classical control parameters of the system. Dynamical phase will be omitted bellow due to the suitable choice of the zero energy level. We shall consider the single subspace (no energy level crossings are assumed).

For optical holonomic quantum computer (as well as for ion trap HQC) one-qubit unitary transformations are given as a sequence of single-mode squeezing and displacing operations

$U = D(\eta)S(\nu)$. Here:

$$S(\nu) = \exp(\nu a^+ a^+ - \bar{\nu} a a), \quad (3)$$

$$D(\eta) = \exp(\eta a^+ - \bar{\eta} a)$$

denotes single-mode squeezing and displacing operators respectively, $\nu = r_1 e^{i\theta_1}$ and $\eta = x + iy$ are corresponding complex control parameters, a and a^+ are annihilation and creation operators. The line over the parameters denotes complex conjugate quantities. The full set of the connection as well as the field strength matrix components can be found in Refs. [3, 4]. In this Letter we consider loops belonging to the planes $(x, r_1)|_{\theta_1=0}$ and $(y, r_1)|_{\theta_1=0}$ only. Corresponding field strength components are [3]:

$$F_{xr_1}|_{\theta_1=0} = -2i\sigma_y \exp(-2r_1), \quad (4)$$

$$F_{yr_1}|_{\theta_1=0} = -2i\sigma_x \exp(2r_1).$$

Here σ_x and σ_y are Pauli matrices. The corresponding holonomies for the loops $C_I \in (x, r_1)_{\theta_1=0}$ and $C_{II} \in (y, r_1)_{\theta_1=0}$ are given by [3]:

$$\Gamma(C_I) = \exp(-i\sigma_y \Sigma_I), \quad \Sigma_I = \int_{S(C_I)} dx dr_1 2e^{-2r_1}, \quad (5)$$

$$\Gamma(C_{II}) = \exp(-i\sigma_x \Sigma_{II}), \quad \Sigma_{II} = \int_{S(C_{II})} dy dr_1 2e^{2r_1},$$

where $S(C_{I,II})$ are the regions in the planes $(x, r_1)|_{\theta_1=0}$ and $(y, r_1)|_{\theta_1=0}$ enclosed by the loops C_I and C_{II} .

Hadamard gate is widely used in various quantum algorithms, for example in quantum Fourier transform, for more details see [21]. It is given as follows:

$$H_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (6)$$

We can obtain it by the two successive y and x rotations on $\pi/4$ and $\pi/2$ respectively:

$$-iH_0 = \Gamma(C_{II})_{\Sigma_{II}=\pi/2} \Gamma(C_I)_{\Sigma_I=\pi/4}. \quad (7)$$

The overall phase factor $(-i)$ is not essential for our purposes. From the experimentalist point of view it is more convenient to hold three control parameters fixed and adiabatically vary the fourth parameter. Thus we consider rectangular loops C_I and C_{II} with its sides to be parallel to the coordinate axes. For C_I these sides are given by the lines $r_1 = 0$, $x = b_x$, $r_1 = d_x$, $x = a_x$. Here a_x and b_x can be chosen arbitrary and the lengths of the rectangle's sides parallel to the x axis are $l_x = b_x - a_x$. In the case of the gate considered d_x is given by the following expression:

$$d_x = -\frac{1}{2} \ln \left(1 - \frac{\pi}{4l_x} \right). \quad (8)$$

It immediately follows that $l_x > \pi/4$. In the same way we set the loop C_{II} as the rectangle composed by the lines $r_1 = 0$, $y = b_y$, $r_1 = d_y$, $y = a_y$ and find that:

$$d_y = \frac{1}{2} \ln \left(1 + \frac{\pi}{2l_y} \right), \quad (9)$$

where $l_y = b_y - a_y$. To hold zero value of the squeezing control parameter is more easy from the experimental point of view than non-zero one. One encounters problems when trying to keep non-zero squeezing parameter and simultaneously adiabatically change the displacing parameter as well as vice versa. In this Letter we restrict ourselves by the errors in the single-mode squeezing parameter control.

To take into account the errors in assignment of the single-mode squeezing parameter r_1 we have to replace d_x by $d_x + \delta r_x(x)$ and d_y by $d_y + \delta r_y(y)$. In this case we obtain that parameters Σ_I and Σ_{II} entering into the expressions (5) are replaced by:

$$\begin{aligned}\Sigma'_I &= \Sigma_I + e^{-2d_x} \int_{a_x}^{b_x} dx \left(1 - e^{-2\delta r_x}\right) = \Sigma_I + \delta\Sigma_I, \\ \Sigma'_{II} &= \Sigma_{II} + e^{2d_y} \int_{a_y}^{b_y} dy \left(e^{2\delta r_y} - 1\right) = \Sigma_{II} + \delta\Sigma_{II}.\end{aligned}\tag{10}$$

Therefore the perturbed Hadamard gate is given by the following expression:

$$-iH = \Gamma(C_{II})|\Sigma'_{II}\rangle\Gamma(C_I)|\Sigma'_I\rangle.\tag{11}$$

Using (5) and (10) we obtain:

$$\begin{aligned}-iH &= -\frac{1}{\sqrt{2}}(\cos\delta\Sigma_I - \sin\delta\Sigma_I)(I \sin\delta\Sigma_{II} + i\sigma_x \cos\delta\Sigma_{II}) - \\ &\quad -\frac{i}{\sqrt{2}}(\cos\delta\Sigma_I + \sin\delta\Sigma_I)(\sigma_z \cos\delta\Sigma_{II} - \sigma_y \sin\delta\Sigma_{II}),\end{aligned}\tag{12}$$

where σ_x , σ_y and σ_z are Pauli matrixes and I is 2×2 identity matrix. Fidelity of the Hadamard gate determining the gate stability with respect to the errors in the control of the single-mode squeezing parameter r_1 is:

$$f_j = \sqrt{|\langle j|iH_0^+(-iH)|j\rangle|^2}, \quad j = 0, 1.\tag{13}$$

Here $|0\rangle$ and $|1\rangle$ are the basis vectors of the qubit. Substituting expressions (6) and (12) into formulae (13) we obtain:

$$f = |\cos\delta\Sigma_I|.\tag{14}$$

Here we see that fidelity does not depend on j and $f_j \equiv f$. As well it is evident that fidelity does not depend on errors made in the $(y, r_1)_{\theta_1=0}$ plane. The reason is that the corresponding x -rotation up to the overall phase factor is just the classical not-gate.

Using expressions (10), (14) and assuming that the control error $\delta r_x(x)$ much less than unity for all x and have the zero average value at the line segment $[a_x, b_x]$ we find:

$$f \simeq \left| \cos \left[\langle \delta r_x^2 \rangle \left(2l_x - \frac{\pi}{2} \right) \right] \right| \simeq 1 - \left(\langle \delta r_x^2 \rangle \right)^2 \left[l_x \sqrt{2} - \frac{\pi}{2\sqrt{2}} \right]^2,\tag{15}$$

where

$$\langle \delta r_x^2 \rangle = \frac{1}{l_x} \int_{a_x}^{b_x} \delta r_x^2(x) dx.\tag{16}$$

Since $l_x > \pi/4$ fidelity $f = 1$ at $l_x = \pi/4$ and equal or less than unity for all over values of the parameters as it should be. As well fidelity is equal to unity for $l_x^{(n)} = \pi/4 + \pi n / (2 \langle \delta r_x^2 \rangle)$

with integer $n > 0$. We believe that this result stems from the fact that we have restricted ourselves by the errors in the squeezing parameter control only. If we take into account errors in assignment of the over control parameters, especially fluctuations of displacing parameters x and y while the squeezing parameter is being adiabatically changed, fidelity of the Hadamard gate will be less than unity for all $l_x^{(n)}$, $n > 0$. However, there are reasons to believe that local maxima at these points will still remain. To clarify it is the task for the further investigations. As well we would like to note the cancellation of the squeezing control errors up to the fourth order on their magnitude that is obviously follows from the expression (15) where we took into account that linear terms were absent in the cosine Taylor expansion. Thus, in this Letter we have analytically derived the expression for the fidelity determining the Hadamard gate stability with respect to the errors in the single-mode squeezing parameter control. We have demonstrated the cancellation of the control errors up to the fourth order on their magnitude under the restrictions and conditions stated above.

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